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Shock waves for discrete velocity nonconservative (except mass) models

H Cornille

Service de Physique Théorique, CE Saclay, F-91191 Gif-sur-Yvette, France

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Abstract. Extended discrete kinetic theory (that which we call nonconservative) including sources, sinks, the creation and annihilation of test particles and inelastic scattering etc added to the elastic collisions, was first introduced by Boffi and Spiga. The mass conservation law (or momentum, energy) becomes, by adding polynomials of the mass (or densities), nonconservative. There exist linear and quadratic nonconservative models for which travelling waves were recently found. In order to test the travelling waves as shock waves (Whitham–Lax criteria and shock inequalities) we consider an intermediate class of models where *in the nonconservative models we restrict the parameters such that the mass conservation law is retained.* We find rarefactive (with mass and pressure decreasing) shocks. *Firstly, for a large class of models*, from the conservative mass relation and the modified momentum (including nonconservative terms) we prove that *only rarefactive shocks can satisfy the shock inequalities*. *Secondly, for particular models*: the two- and three-dimensional Broadwell models, the hexagonal $6v_i$ model and $8v_i$, $9v_i$ squares models (without and with rest particle) *we construct explicit travelling solutions* which result from the compatibility between different scalar Riccati equations. We find a condition such that the Lax criterion and the shock inequalities are satisfied, while we numerically check the Whitham criterion. We also find that a criterion for overshoots of the internal energy, established previously for conservative models, still works for the nonconservative ones.

1. Introduction

Extended discrete kinetic theory (that which we call nonconservative), introduced by Boffi and Spiga, has been extensively studied [1]. They add to elastic collisions a background medium, external sources and sinks, effects of absorption and generation due to inelastic scattering etc. A great difference, with conservative discrete velocity models (DVMs), is that the *conservation laws are modified* by including nonconservative terms: for instance, by adding polynomials functions of the densities, either linear [2] or quadratic [3]. Recently, exact travelling wave solutions and (1 + 1)-dimensional solutions were found [4].

For the densities $N_i(z)$ travelling waves with the variable $z = x - \xi t$ (wave speed ξ, space x, time t) of nonconservative DVMs with binary collisions and quadratic nonconservative terms ([4] also deals with linear terms), the exact solutions were obtained from compatible different scalar Riccati equations number equal to the independent densities. For conservative or nonconservative (except mass) DVMs this number is smaller because we substract the number of linear conservation laws (invariants). A difficulty for the conservative DVMs is that this number can be larger than the number of physical invariants (mass, momentum and energy). The nonphysical invariants are called 'spurious invariants'. In contrast, for nonconservative

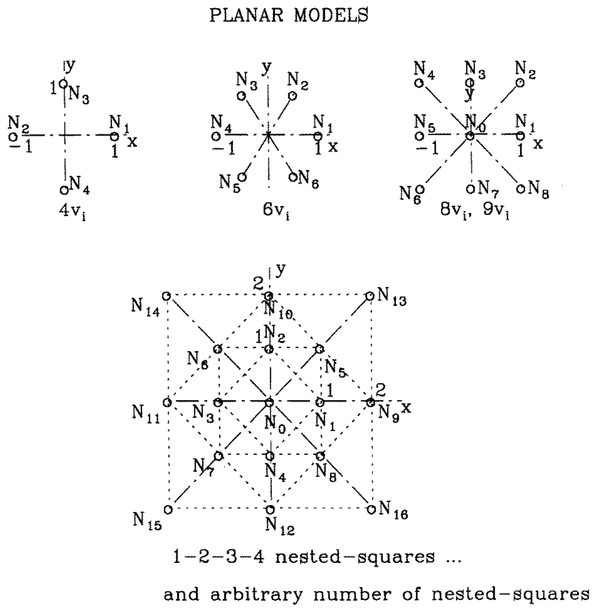

Figure 1.

DVMs with only the mass conservation invariant, we have no spurious invariant. The travelling densities $N_i(z)$ waves and the solutions of scalar Riccati equations of the type (see section 4):

$$N_i(z) = n_{0i} + (n_{si} - n_{0i})/[1 + e^{\gamma z}] \qquad \gamma,\ n_{0i},\ n_{si} \text{ constants} \tag{1.1}$$

remain almost constant for $|z|$ large (see figures 2–4), but their two asymptotic states are either n_{0i} or n_{si} at $z = \pm\infty$. Similarly, for the macroscopic mass $\rho(z)$ and the momentum $J(z)$ along the x-axis, the asymptotic states at $z = \pm\infty$ are either ρ_0, j_0 or ρ_s, j_s. For shock waves we must first consider the direction of the shock (from the upstream to the downstream states) and determine the asymptotic states which are either the upstream (satisfying a supersonic inequality) or the downstream state (satisfying a subsonic inequality). If the direction of the shock is positive, then the upstream (downstream) state is at $z = -\infty$ ($z = +\infty$). In contrast, for a negative direction the upstream (downstream) states are at $z = +\infty$ ($z = -\infty$).

In section 2, see figure 1, we define, in the $d = 2$ plane with two coordinates (x, y), a class of models with velocities $\vec{v}_i = (x = e_i, y = y_i)$ (in addition, only the Broadwell $d = 3$ dimensional is considered), and briefly recall previous results and the different formalisms used in the paper. The planar models are such that for any $\vec{v}_i$ exists its opposite, $\vec{v}_j$, with $\vec{v}_i + \vec{v}_j = (0, 0)$ or for the projections e_i, e_j along the x-axis we have $e_i + e_j = 0$. In particular, for each $\vec{v}_i = (e_i, 0)$ there is $\vec{v}_j = (-e_i, 0)$. Furthermore, for every $\vec{v}_i$ there is $\vec{v}_k$ such that the sum $\vec{v}_i + \vec{v}_k = 2(e_i, 0)$ is along the x-axis. As an illustration some models are presented in figure 1 where, for brevity, we substitute the densities N_i to the velocities. The N_i densities depend on (x, y, t) but for travelling waves along the x-axis, the y coordinate disappears and they are only $(z = x - \xi t)$ dependent. We only write the evolution equations for the (x, t)-dependent solutions and write for the conservative, the nonconservative and the nonconservative (except mass) models the evolution equations for both the microscopic $N_i(z)$ and macroscopic $\rho(z)$, $J(z)$ quantities. We define the nonconservative terms and the restrictions when we re-introduce the mass conservation. Later we define the asymptotic states, the Lax–Whitham criteria [5] and the shock inequalities. For conservative DVMs, results have been obtained in the general case [6, 7] and for other particular models with specific collisions [8]. As quoted by Whitham [5]:

> In some wave propagation, different levels of approximation to the govering equations lead to different types of wave motion. A related effect is that the propagation speeds defined by the highest derivatives may be quite different from the speeds at which the main disturbance travels. Then the questions arise to know which sets of waves are dominant.

For shock waves the main disturbance is the sound wave (or characteristic) which corresponds to the lowest-order derivative and the problem is to check whether the waves associated to higher orders are damped when $t \to \infty$. Broadwell was the first to introduce this Whitham criterion for his shock wave. For conservative DVMs general [6] and particular [8] results have been obtained for this criterion, exploiting the fact that the asymptotic states correspond to vanishing elastic collision terms. For nonconservative, except mass models, this is no more true for the asymptotic states and almost all results presented are numerical. The Lax criterion was first introduced in a different context for hyperbolic systems of conservation laws and later [6–8] for conservative DVMs. Naming $\xi_{\pm\infty}$ the sound waves (or characteristics) associated to the asymptotic states at $z = \pm\infty$ and ξ the shock speed, for the Lax criterion we must satisfy $\xi_{+\infty} < \xi < \xi_{-\infty}$. As we shall see here analytically (lemma 1) and numerically (tables associated with figures 2–4), it results from both a common direction for the shock and sound waves and the supersonic and subsonic inequalities.

In section 3 for the class of models defined in section 2 we obtain general results for the nonconservative (except mass) models. We determine analytically the sound waves associated to the asymptotic states and prove that only rarefactive shock can exist. This is in contrast with conservative models for which both compressive and rarefactive shocks have been found. For instance, for the conservative Broadwell models we prove (appendix A1) that only compressive shocks exist. This illustrates the great difference between conservative and nonconservative (except mass) shock solutions.

In section 4 we recall the determination of travelling waves coming from the compatibility between different scalar Riccati equations. In sections 5–7 we study the Broadwell, hexagonal and square models.

Throughout the following, for any Z_i, z_i quantities we define

$$Z^{\pm}_{i,j} = Z_i \pm Z_j \qquad z^{\pm}_{i,j} = z_i \pm z_j \qquad Z_{i,j,\ldots,p} = Z_i + Z_j + \cdots Z_p \qquad z_{i,j,\ldots,p} = z_i + z_j + \cdots z_p. \tag{1.2}$$

2. Models, evolution equations, Lax–Whitham criteria, shock inequalities

2.1. Models with opposite discrete velocities in the (x, y) plane

In figure 1 we present some planar $d = 2$ models with two coordinates for the velocities which are symmetric with respect to both the x- and y-axes: the $4v_i$ Broadwell with velocity coordinates $(\pm1, 0)$, $(0, \pm1)$, the $6v_i$ hexagonal model with velocities $(\pm1, 0)$, $(\pm\frac{1}{2}, \pm\sqrt{3}/2)$, the $8v_i$ square model without a rest particle $(\pm1, \pm1)$, $(\pm1, 0)$, $(0, \pm1)$ and adding a rest particle $(0, 0)$ the $9v_i$, and finally the $4pv_i$ nested p-square model. We will consider only one $d = 3$ Broadwell model with three coordinates for the velocities: $(\pm1, 0, 0)$, $(0, \pm1, 0)$, $(0, 0, \pm1)$.

The one spatial dimensional x, t solutions, depending only on the spatial coordinate x, have independent densities $N_i(x, t), i = 1, 2, \ldots$. For the planar models those associated to velocities (x_i, y_i) with the same $x_i = e_i$ value but opposite y_i values are equal. The N_i

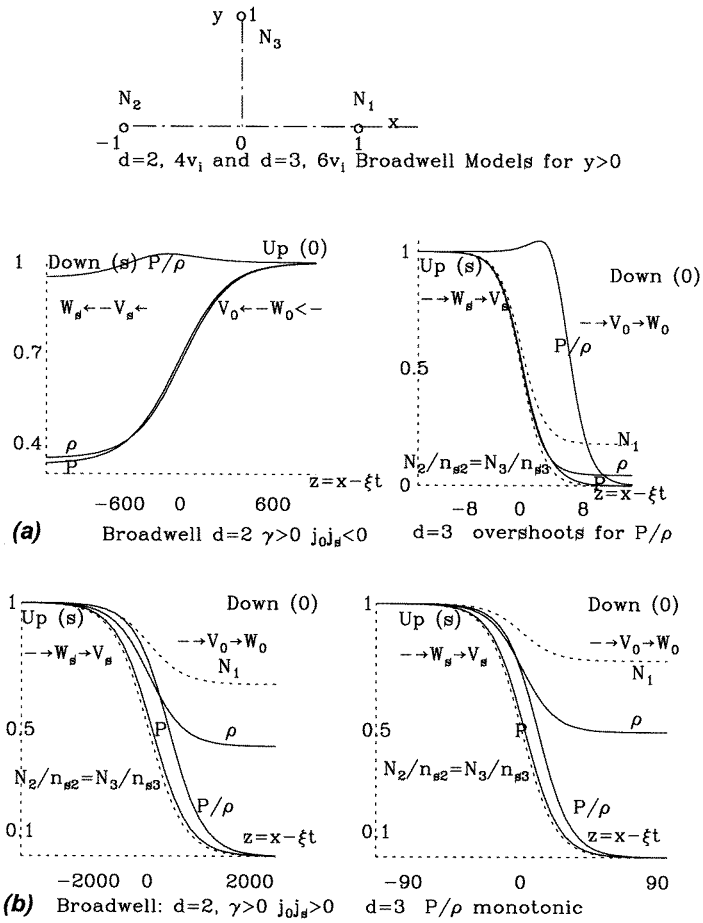

Figure 2.

associated to velocities ($x_i = e_i, y_i = 0$) along the x-axis have multiplicity $\alpha_i = 1$ and for any N_i exists another N_j with a velocity $e_j = -e_i$ along the x-axis. In contrast those with velocities ($x_i = e_i, \pm y_i \neq 0$) are equal with multiplicity $\alpha_i = 2$. Finally, for any velocity $\vec{v}_i$ with an x-axis projection $e_i \neq 0$ there exists another $\vec{v}_j$ such that $\vec{v}_i + \vec{v}_j = (0, 0)$, $e_j + e_i = 0$ and another $\vec{v}_k$ with $\vec{v}_i + \vec{v}_k = 2(e_i, 0)$. Only for the $d = 3$ Broadwell model, do the densities associated to $(\pm 1, 0, 0)$ have $\alpha_i = 1$, while the four other densities with projections on the x-axis $x_i = 0$ correspond to one density with $\alpha_i = 4$.

For the one-dimensional solutions $N_i(x, t)$ we define the macroscopic total mass ρ, the momentum J along the x-axis and the elastic binary collisions Col_{N_i}:

$$\rho = \sum \alpha_j N_j \qquad J = \sum \alpha_i e_i N_i \qquad \mathrm{Col}_{N_i} = \sum_{j,k,l} a_{i,j}^{k,l} (N_k N_l - N_j N_i) \tag{2.1}$$

where $a_{i,j}^{k,l}$ is the transition probability of the collision $(\vec{v}_i, \vec{v}_j) \to (\vec{v}_k, \vec{v}_l)$. For the square Broadwell model we have $\rho = N_{1,2}^{+} + 2N_3$, $J = N_{1,2}^{-}$, with multiplicity 1 for N_1, N_2 and $e_i = \pm 1$ while $e_3 = 0$, $\alpha_3 = 2$ for $N_3 = N_4$. For the hexagonal model we have $\rho = N_{1,4}^{+} + 2N_{2,3}^{+}$, $J = N_{1,4}^{-} + N_{2,3}^{-}$, and see $N_2 = N_6, N_3 = N_5$ with $\alpha_i = 2$ and $e_i = \pm\frac{1}{2}$ while $e_i = \pm 1$ and $\alpha_i = 1$ for N_1, N_4. For the squares $\rho = N_{0,1,5} + 2N_{2,3,4}$, $J = N_{1,5}^{-} + 2N_{2,4}^{-}$,

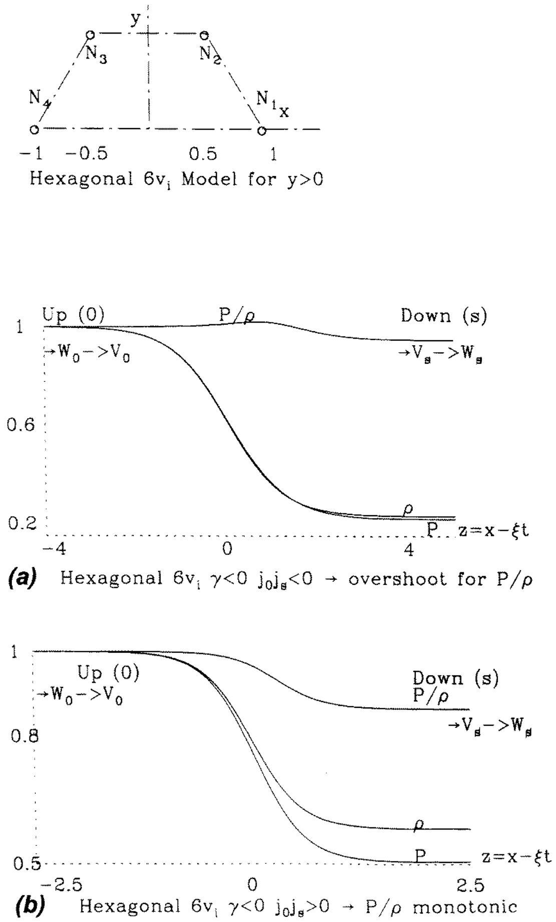

Figure 3.

we see $N_2 = N_8, N_3 = N_7, N_4 = N_5$ with $\alpha_i = 2$ and $e_i = \pm 1, 0$. Of course for the rest particle N_0 we have $\alpha_0 = 1$ and $e_0 = 0$. For the qth nested square with densities $N_i = N_{4(q-1)+j}$, $j = 1, 2, 3, 4$ we have for q odd $\alpha_i = 1$ and $e_i = \pm 2^{(q-1)/2}$ or $\alpha_i = 2$ and $e_i = 0$. For q even we always have $\alpha_i = 2$ and $e_i = \pm 2^{q/2-1}$. We can also construct other nested-polygonial models like the nested hexagons with similar properties. For the $d = 3$ Broadwell model with opposite velocities ± 1 along the third z-axis we see that for the x-dependent solution the only change is for N_3 with $\alpha_3 = 4$ and $\rho = N_{1,2}^{+} + 4N_3$. Other $d = 3$ models exist with a similar structure and $\alpha_i = 2(d-1)$ but it can be eight. For brevity they are not considered.

Consequently, for all these models: $\sum e_i = \sum e_i \alpha_i = 0$.

2.2. Evolution equations for the conservative, nonconservative and nonconservative (except mass) models

First, for the *conservative* models we write the $N_i(z = x - \xi t)$ evolution equations:

$$L_i = (\partial_t + e_i \partial_x) N_i = \partial_z(-\xi + e_i) N_i = R_i = \mathrm{Col}_{N_i}. \tag{2.2a}$$

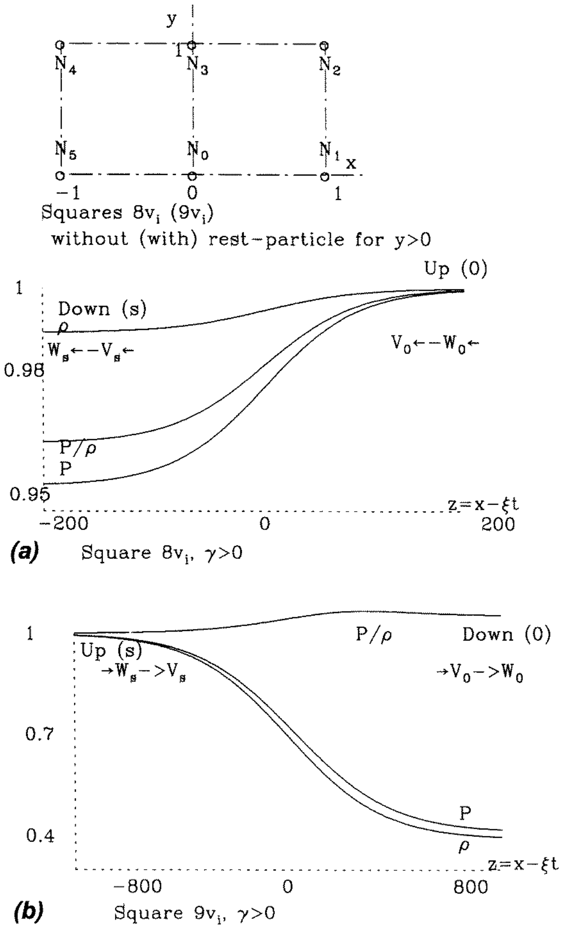

Figure 4.

We obtain, mulplying by e_i, $\alpha_i e_i$ and summing, both the ρ and J equations:

$$L_\rho = \partial_t\rho + \partial_x J = \partial_z(-\xi\rho + J) = R_\rho = 0 \qquad \sum \alpha_i \mathrm{Col}_{N_i} = 0 \tag{2.2b}$$

$$L_J = \partial_t J + \partial_x \rho_J = \partial_z(-\xi J + \rho_J) = R_J = 0 \qquad \sum \alpha_i e_i \mathrm{Col}_{N_i} = 0 \quad \rho_J = \sum e_i^2 \alpha_i N_i. \tag{2.2c}$$

Secondly, for the *nonconservative* models we write the N_i evolution equations and still, multiplying by $e_i, \alpha_i e_i$ and summing, the mass and momentum along the x-axis relations which are not conservative:

$$L_i = (\partial_t + e_i\partial_x)N_i = \partial_z(-\xi + e_i)N_i = R_i = \mathrm{Col}_{N_i} + \kappa_i(\alpha\rho^2 + \eta\rho) - N_i(\beta\rho + \epsilon) + S_i \tag{2.3a}$$

$$L_\rho = \partial_t\rho + \partial_x J = \partial_z(-\xi\rho + J) = R_\rho = (\alpha - \beta)\rho^2 + (\eta - \epsilon)\rho + S_\rho \qquad S_\rho := \sum \alpha_j S_j \tag{2.3b}$$

$$L_J = \partial_t J + \partial_x \rho_J = \partial_z(-\xi J + \rho_J) = R_J = \kappa_J[\alpha\rho^2 + \eta\rho] + S_J - J(\beta\rho + \epsilon)$$
$$\kappa_J = \sum e_i \alpha_i \kappa_i \qquad S_J = \sum e_i \alpha_i S_i. \tag{2.3c}$$

Col_{N_i} are the only terms present in the conservative models. The others are the nonconservative terms [2, 3]: $\beta \geqslant 0$, $\epsilon \geqslant 0$ ($\alpha \geqslant 0$, $\eta \geqslant 0$) for the destruction (creation) of test particles as a result of *inelastic collisions which can be quadratic or linear in the densities*. The $\kappa_i \in (0, 1)$ with $\sum \alpha_i \kappa_i = 1$ represent the fractions of secondary particles generated with velocities e_j along the x-axis. As usual, $S_i > 0$ (<0) are constants associated to the *external sources (sinks)*.

Thirdly, we consider nonconservative (except mass) DVMs. In order to establish some link between conservative and nonconservative DVMs, we keep the mass conservation relation $L_\rho = R_\rho$ with $R_\rho \equiv 0$ or $\alpha = \beta$, $\eta = \epsilon$, $S_\rho = 0$ and obtain an intermediate class of models. The amount of test particles created or annihilated as well as the sum of sources and sinks is globally zero. Of course the momentum and energy relations are not conserved relations, and the fraction of secondary particles κ_i is not modified. As a consequence we will try to extend important properties of the conservative models for shock waves, namely to check the Whitham–Lax criteria and the shock inequalities. With ρ, J, still defined in (2.1), Col_{N_i} satisfying (2.1)–(2.2*c*) and ρ_J, κ_J, S_J in (2.2*c*)–(2.3*c*) we rewrite the equations for the densities with elastic collisions and nonconservative terms, the mass conservation (without nonconservative terms) and the momentum relation (with nonconservative terms), but without elastic collisions.

$$\text{densities:}\quad (\partial_t + e_i\partial_x)N_i = \partial_z(-\xi + e_i)N_i = \mathrm{Col}_{N_i} + (\alpha\rho + \eta)(\kappa_i\rho - N_i) + S_i \tag{2.4a}$$

$$\text{mass:}\quad \partial_t\rho + \partial_x J = \partial_z(-\xi\rho + J) = 0 \qquad S_\rho = 0 \qquad \beta = \alpha \qquad \epsilon = \eta \tag{2.4b}$$

$$\text{momentum:}\quad \partial_t J + \partial_x \rho_J = \partial_z(-\xi J + \rho_J) = (\alpha\rho + \eta)(\kappa_J\rho - J) + S_J. \tag{2.4c}$$

2.3. Asymptotic states, Lax–Whitham criteria, shock inequalities

In the following we consider DVMs satisfying only the mass conservation (2.4*a*)–(2.4*c*). For travelling waves $N_i(z = x - \xi t)$ with wave speed ξ we assume, from the vanishing of the rhs in (2.4*a*), the existence of two asymptotic states (0) and (s) when $|z| \to \infty$:

- (0) with n_{0i}, $\rho_0 = \sum \alpha_i n_{0i}$, $j_0 = \sum \alpha_i e_i n_{0i}$;
- (s) with $n_{si} = n_{0i} + n_i$, $\rho_s = \sum \alpha_i n_{si} = \rho_0 + \rho_1$, $j_s = \sum e_i \alpha_i n_{si} = j_0 + j_1$.

$$\text{Densities:}\quad -S_i = \mathrm{Col}_{n_{0i}} + (\alpha\rho_0 + \eta)(\kappa_i\rho_0 - n_{0i}) = \mathrm{Col}_{n_{si}} + (a\rho_s + \eta)(\kappa_i\rho_s - n_{si}). \tag{2.5a}$$

The mass conservation $\partial_z(-\xi\rho + J) = 0$ can be integrated leading to the jump relation:

$$\text{mass:}\quad \text{def:}\quad \begin{array}{l} V_0 = j_0/\rho_0 - \xi \qquad V_s = j_s/\rho_s - \xi \to V_0\rho_0 = V_s\rho_s \\ V_0 V_s > 0 \quad j_1 = \xi\rho_1. \end{array} \tag{2.5b}$$

The vanishing of the rhs of the momentum (2.4*c*) where we introduce (2.5*b*) remains:

$$\begin{array}{l} \text{momentum:}\quad -S_J = (\alpha\rho_s + \eta)(\kappa_J\rho_s - j_s) = (\alpha\rho_0 + \eta)(\kappa_J\rho_0 - j_0) \\ \qquad \to (\alpha\rho_s + \eta)(\kappa_J - \xi) = \alpha(j_0 - \kappa_J\rho_0) \qquad \text{and} \qquad \rho_0 \rightleftarrows \rho_s,\ j_0 \rightleftarrows j_s. \end{array} \tag{2.5c}$$

The last (2.5*c*) relation, mixing ξ, ρ_0, ρ_s and the nonconservative terms, is very important.

Whitham criterion. Linearizing around an asymptotic state we get a sum of linear differential operators of a different order, the lower giving the characteristics (sound waves). For two successive operators Whitham has given a criterion such that the lower is dominant when $t \to \infty$:

$$\left[\prod_{j=1}^{p+1}(\partial_t + c_j\partial_x) + \lambda\prod_{j=1}^{p}(\partial_t + a_j\partial_x)\right]\Phi(x, t) = 0 \qquad \lambda > 0 \quad c_1 > a_1 > c_2 \cdots > a_p > c_{p+1}. \tag{2.6}$$

In order to understand the difficulty of a complete proof of the different Whitham criteria, let us consider our nonconservative (except mass) models where the lowest operator is of the first order with, for instance, characteristic speed ξ_0 at the (0) asymptotic state. The next order operator is second order with wave speeds $c_1 > c_2$. We must prove that $c_1 > \xi_0 > c_2$. The next one is of third order with speeds $d_1 > d_2 > d_3$ and we must check the interlacing properties $d_1 > c_1 > d_2 > c_2 > d_3$ and so on. In general we have only verified this numerically. In contrast, for conservative DVMs the asymptotic states correspond to vanishing elastic collision terms and both general [6, 7] (binary collisions), and particular [8] results (multiple collisions) have been obtained.

Lax criterion. Let ξ be the shock speed, and ξ_0, ξ_s the speeds of the sound waves or characteristics associated to the (0), (s) states and compare these with the shock speed ξ. If we have $\xi_0 \leqslant \xi \leqslant \xi_s$ then the (0) state must be on the right or on the left (conversely for the (s) state). For our travelling waves right and left means asymptotic states at $\pm\infty$. So we define as $\xi_{\pm\infty}$ the characteristics associated to the states at $\pm\infty$ (which are either ξ_0, or ξ_s). For the Lax criterion [5–8] they must satisfy: $\xi_\infty < \xi < \xi_{-\infty}$ (with the same index if there exist more than one ξ_0, ξ_s).

Shock inequalities. To the shock waves V_0, V_s satisfying the jump condition (2.5) with $V_0 V_s > 0$ we associate the waves $W_0 = j_0/\rho_0 - \xi_0$, $W_s = j_s/\rho_s - \xi_s$ at the (0), (s) states and they must satisfy the shock inequalities. The direction of the shock (from the upstream to the downstream state) is given by the common sign of V_0, V_s. If it is positive then the upstream is at $-\infty$ and the downstream at $+\infty$ (the converse if it is negative). If the (0) state is in an upstream state we must have the supersonic inequality $|V_0| > |W_0|$ while if it is downstream, the subsonic inequality $|V_0| < |W_0|$ (similarly for the (s) state). We redefine these shock inequalities from the unknown states at $\pm\infty$. Let $\rho_\pm$, $j_\pm$, $\xi_\pm$ be the mass, momentum and sound-wave speed at $z = \pm\infty$ and define $V_\pm = j_\pm/\rho_\pm - \xi$, $W_\pm = j_\pm/\rho_\pm - \xi_\pm$ ($V_\pm$, $W_\pm$ can be either V_0, W_0 or V_s, W_s). For an upstream state at $\pm\infty$ we must satisfy the supersonic inequality $|V_\pm| > |W_\pm|$ and for a downstream state at $\pm\infty$ the subsonic inequality $|V_\pm| < |W_\pm|$.

Assumptions which are usual. From the jump relation $V_0\rho_0 = V_s\rho_s$ with $\rho_0 > 0, \rho_s > 0$ the direction of the shock is given by the common sign of V_0, V_s. Let us assume $V_0 > 0$, a positive direction of the shock and the (0) state at $z = -\infty$. The (0) state is the upstream state and we must verify a supersonic inequality. If $W_0 < 0$, the shock wave and the sound-wave perturbation propagate in opposite directions and, as stated in the literature [6], physically we cannot say that the shock is supersonic. Similarly at the downstream if $V_s > 0$, $W_s < 0$. So we assume that the sound waves W_0, W_s also have the shock-wave direction, the subsonic and supersonic inequalities for both the shock and sound waves.

Lemma 1. *If the shock inequalities are satisfied then the Lax criterion is also satisfied.*

For the proofs we have two cases: $V_0 > 0, < 0$ with a positive and negative direction of the shock.

(1) $V_0 > 0$, then the upstream is at $-\infty$ with a supersonic inequality $V_- = j_-/\rho_- - \xi > W_- = j_-/\rho_- - \xi_-$ or $\xi < \xi_-$, and the downstream at $+\infty$ with a subsonic inequality $V_+ < W_+$ or $\xi_+ < \xi$ and the Lax criterion is satisfied $\xi_+ < \xi < \xi_-$.

(2) $V_0 < 0$, then the upstream is at $+\infty$ with a supersonic inequality $-V_+ = -j_+/\rho_+ + \xi > -W_+ = -j_+/\rho_+ + \xi_+$ or $\xi_+ < \xi$, and the downstream at $-\infty$ with a subsonic inequality $-W_- > V_-$ or $\xi < \xi_-$ with the Lax criterion.

So ξ is necessarily in the interval (ξ_0, ξ_s) but we do not know whether $\xi_0 \gtrless \xi_s$.

3. General results without explicit solutions

We will obtain general results for any DVMs (without the construction of explicit solutions) satisfying *only one conservative relation* (the mass conservative relation) and a nonconservative momentum relation of the type (2.4*c*) (as for the nonconservative, except mass, models). First, linearizing around the (0) state we will get for all nonconservative (except mass) DVMs, with equations written in (2.4*a*)–(2.4*c*) (except the two-velocity models [4] not considered here), the same expression for the sound waves ξ_0 (ξ_s) associated to (0), (s):

$$[\rho_0\alpha + \eta][\kappa_J - \xi_0] = \alpha[j_0 - \kappa_J\rho_0] \qquad \text{and} \qquad \xi_0, \rho_0, j_0 \to \xi_s, \rho_s, j_s. \tag{3.1}$$

Second, we will consider only the mass conservation, with the jump relation (2.5*b*), the existence of two asymptotic states (0), (s) satisfying the vanishing relations of the momentum modified relations (2.5*c*), V_0 giving the direction of the shock and the shock inequalities. *We will prove that only rarefactive shocks can exist.*

3.1. Linearization around the (0) state

With the (2.4*b*), (2.4*a*), (2.4*c*) linear and nonlinear system we assume $\rho \simeq \rho_0[1 + X_\rho(z)]$, $J \simeq j_0[1 + X_J(z)]$ and $N_i = n_{0i}[1 + X_i(z)]$ with $z = x - \zeta t$, respectively. For a model with p independent densities we linearize with the system of X_ρ, X_j and $(p-2)$ X_i and take the determinant. From (2.4*b*) the first row, for the mass conservation, contains only ∂_t and ∂_x. Consequently, the lowest order operator is of first order and the associated ζ polynomial is linear. From (2.4*c*) the second row, for the momentum, has constant terms given from the rhs only for X_ρ, X_J. The remaining terms contain derivatives ∂_x, ∂_t. For the determination of the characteristics $\zeta = \xi_0$ it is sufficient to consider constant terms in the rows different from the first one, corresponding to (2.4*a*)–(2.4*c*). In particular, in the second row for the momentum, except for the first and the second column, all other terms are zero. This is the main reason why we obtain the sound waves only from the mass and momentum relations where for $z = z_0 = x - \xi_0 t$ we have $\partial_t = -\xi_0\partial_{z_0}$, $\partial_x = \partial_{z_0}$:

$$\begin{vmatrix} \partial_t & \partial_x & 0 & 0\ldots & 0 \\ -A_\rho & A_J & 0 & 0\ldots & 0 \\ A_{31} & A_{32} & A_{33} & \ldots & A_{3p} \\ \cdot & \cdot & \cdot & \ldots & \cdot \\ A_{p1} & A_{p2} & A_{p3} & \ldots & A_{pp} \end{vmatrix} = 0 = (\partial_t A_J + \partial_x A_\rho) \begin{vmatrix} A_{33} & \ldots & A_{3p} \\ \cdot & \ldots & \ldots \\ A_{p3} & \ldots & A_{pp} \end{vmatrix} = 0 \tag{3.2}$$

$$A_J = \alpha\rho_0 + \eta > 0 \qquad A_\rho = (2\alpha\rho_0 + \eta)\kappa_J - \alpha j_0 \to \xi_0 = A_\rho/A_J \tag{3.3}$$

and we recover the (3.1) results for the sound waves ξ_0, ρ_0, j_0 and ξ_s, ρ_s, j_s.

3.2. Rarefactive shocks for the nonconservative (except mass) models

We recall V_0, V_s written in (2.5*b*) and notice that in (3.1) and in the last (2.5*c*) relation the rhs are the same. This means that direct relations exist between the shock-wave speed ξ and the sound-wave speeds ξ_0, ξ_s:

$$[\alpha\rho_s + \eta][\kappa_J - \xi] = \alpha(j_0 - \kappa_J\rho_0) = \alpha\rho_0[V_0 - (\kappa_J - \xi)] = [\alpha\rho_0 + \eta][\kappa_J - \xi_0] \tag{3.4a}$$

$$[\alpha\rho_0 + \eta][\kappa_J - \xi] = \alpha(j_s - \kappa_J\rho_s) = \alpha\rho_s[V_s - (\kappa_J - \xi)] = [\alpha\rho_s + \eta][\kappa_J - \xi_s]. \tag{3.4b}$$

For the linear nonconservative models $\alpha = 0$, $\eta \neq 0$, from (3.4*a*), (3.4*b*) we get $\kappa_J = \xi = \xi_0 = \xi_s$, $V_0 = W_0$, $V_s = W_s$ without possible shock inequalities. In the following we assume $\alpha \neq 0$.

Lemma 1(i). *From* (3.4*a*), (3.4*b*) *we get* $V_0(\kappa_J - \xi) > 0$, $V_s(\kappa_J - \xi) > 0$ *and verify* $\xi \in (\xi_0, \xi_s)$.

For the proof of the first result we get from the positivity of $\alpha, \eta, \rho_0, \rho_s$ and (3.4*a*): $(\alpha\rho_s + \eta)/\alpha\rho_0) = V_0/(\kappa_J - \xi) - 1 > 0$ and similarly from (3.4*b*). For the second result we subtract the first and last terms in both (3.4*a*), (3.4*b*) with positive $\alpha, \eta, \rho_0, \rho_s$:

$$\alpha\rho_1(\kappa_J - \xi) = (\alpha\rho_0 + \eta)(\xi - \xi_0) = (\alpha\rho_s + \eta)(\xi_s - \xi). \tag{3.4c}$$

Theorem 1. *The shocks are rarefactive for nonconservative (except mass) DVMs.*

For the proofs we must show that ρ is larger upstream than downstream. We have four different cases: $\rho_1 = \rho_s - \rho_0 \gtrless 0$ and $\kappa_J \gtrless \xi$.

(1) $\kappa_J > \xi$ giving $V_0 > 0$ from lemma 1(i) and a positive direction with the upstream (downstream) states at $z = \mp\infty$.
- (1.1) $\rho_1 > 0$. From (3.4*a*), (3.4*b*) we deduce $\kappa_J - \xi_s < \kappa_J - \xi < \kappa_J - \xi_0$, $\xi_0 < \xi < \xi_s \to V_0 - W_0 = \xi_0 - \xi < 0$ or $V_0 < W_0$ with a subsonic (0) downstream state inequality. The upstream state is the (s) state with $\rho_s > \rho_0$ and ρ decreases from the upstream to the downstream states.
- (1.2) $\rho_1 < 0$. We deduce $\xi_s < \xi < \xi_0$, $V_0 > W_0$ with a supersonic (0) state inequality and a mass $\rho_0 > \rho_s$ at the downstream (s) state.

(2) $\kappa_J < \xi$ giving $V_0 < 0$, a negative direction of the shock with the upstream (downstream) states at $z = \pm\infty$.
- (2.1) $\rho_1 > 0$ giving $\rho_s > \rho_0$, $\xi_s < \xi < \xi_0$, $W_0 < V_0 < 0$ and a subsonic downstream (0) state with ρ_0 less than ρ_s at the upstream.
- (2.2) $\rho_1 < 0$ with $\xi_0 < \xi < \xi_s$, $V_0 < W_0 < 0$ and a supersonic (0) upstream state $|V_0| > |W_0|$ inequality with ρ_0 larger than ρ_s downstream.

In order to verify that these possible rarefactive shocks really exist, with decreasing pressure, we must construct explicit solutions and, for instance, verify that the positivity for the densities $n_{0i} > 0$, $n_{s,i} > 0$ are satisfied. This is the goal of the following sections.

4. Travelling waves as compatible scalar Riccati solutions

In [4, 5] we gave the method in order to obtain, from a quasi-linear system $L_i[N_i] = R_i[N_i]$, exact travelling waves $N_i(z = x - \xi t)$ solutions which result from the compatibility between p different scalar Riccati solutions. We briefly recall the method. The L_i are linear differential x, t operators, for instance as in (2.4*a*) $L_i = \partial_t + e_i\partial_x = (-\xi + e_i)\partial_z$, but they can also be a sum of such terms. The $R_i[N_i] = R_i(N_1, N_2, \ldots, N_p)$ are sums of quadratic nonlinear terms in $N_1, N_2 \ldots$ as in (2.4*a*)–(2.4*c*) but they can also contain linear terms and constants.

We assume that all $N_i(z)$ are linear functions of the same $N(z)$ function with two asymptotic states (0): n_{0i} and (s): $n_{si} = n_{0i} + n_i$ corresponding to $R_i[n_{0i}] = R_i[n_{si}] = 0$:

$$N_i(z) = n_{0i} + n_i N(z = x - \xi t) \qquad \rho(z) = \rho_0 + \rho_1 N(z) \qquad J(z) = j_0 + j_1 N(z). \tag{4.1}$$

We substitute (4.1) into $R_i[N_i]$ and from $R_i[n_{0i}] = R_i[n_{0i} + n_i] = 0$ we deduce for $R_i[n_{0i} + n_i] - R_i[n_{0i}] = 0$ a sum of terms quadratic B_i and linear $-B_i$ in $[n_i] = n_1, n_2, \ldots$. We can rewrite the rhs as $R_i = B_i N(N - 1)$. On the other hand, the lhs can be written

$L_i = C_i \mathrm{d}N/\mathrm{d}z$ so that we are reduced to a set $C_i \mathrm{d}N/\mathrm{d}z = B_i N(N-1)$ of scalar Riccati equations which are compatible only if $\gamma_i := B_i/C_i$ give the same constant for all $i = 1, \ldots, p$,

$$\mathrm{d}N/\mathrm{d}z = \gamma_i N(z)(N(z)-1) \qquad \text{conditions: } \gamma = \gamma_1 = \gamma_2 = \cdots \to N(z) = [1 + \mathrm{e}^{\gamma z}]^{-1} \tag{4.2}$$

and $N_i \to n_{0i}, n_{si}$ when $|z| \to \infty$. Applying this method to the travelling (2.4*a*)–(2.4*c*) DVMs, we add to the (2.5*a*)–(2.5*c*) relations for the densities and the momentum relations:

$$\gamma_i(-\xi + e_i)n_i = \mathrm{Col}_{n_i} + \alpha\rho_1(\kappa_i\rho_1 - n_i) \qquad \gamma = \gamma_1 = \gamma_2 \ldots \tag{4.3a}$$

$$\text{def:} \quad \bar{\rho}_J := \sum e_i^2 \alpha_i n_i/\rho_1 \qquad C_{Lax} := \bar{\rho}_J - \xi^2 \qquad \gamma C_{Lax} = \alpha\rho_1(\kappa_J - \xi). \tag{4.3b}$$

We prove that the *Lax criterion and the shock inequalities are satisfied only if* $C_{Lax} > 0$.

Theorem 2. *For nonexplicit scalar Riccati solutions the Lax Criterion is satisfied (not) if* $C_{Lax} > 0$ *(*< 0*) and, from (4.3b),* $\gamma\rho_1(\kappa_J - \xi) > 0$ *if* $C_{Lax} > 0$.

For the proof we notice from (4.2)–(4.3*b*) that $\gamma \gtrless 0 \to (0)$ state at $z = \pm\infty$ and (s) at $z = \mp\infty$, while γC_{Lax} written in (4.3*b*) is equal to the terms written in (3.4*c*):

$$\gamma C_{Lax} = (\alpha\rho_0 + \eta)(\xi - \xi_0) = (\alpha\rho_s + \eta)(\xi_s - \xi). \tag{4.4}$$

We get two cases $\xi_0 \lesseqgtr \xi \lesseqgtr \xi_s$ associated to $\gamma C_{Lax} \gtrless 0$. Consequently with $C_{Lax} > 0, \gamma > 0$ then $\rho_0 = \rho_{+\infty}$ and $\xi_0 < \xi < \xi_s$ while if $\gamma < 0$ then $\rho_s = \rho_{+\infty}$ and $\xi_s < \xi < \xi_0$. In contrast if $C_{Lax} < 0$ we find $\xi_{-\infty} < \xi < \xi_{+\infty}$ in both cases.

In sections 5–7 we explicitly construct compatible travelling solutions for the Broadwell, $6v_i, 8v_i, 9v_i$ nonconservative DVMs and give examples ($C_{Lax} > 0$) with Lax and shock inequalities satisfied. For the Whitham criterion, the sound waves being given with a linear polynomial, *interlacing properties with higher polynomials are deduced numerically.*

5. Shock waves for Broadwell $d = 2, 4v_i$, $d = 3, 6v_i$ models

With three independent densities (cf figure 1) N_1, N_2, N_3 and x-coordinates $e_i = \pm 1, 0$ or equivalently ρ, J, N_3, we have three evolution equations. We write the mass and momentum conservations (2.4*b*), (2.4*c*) without elastic collisions and finally N_3 (multiplicity $2(d-1)$) with collisions. We recall for the mass the jump relation and write the other two as:

$$\begin{aligned} &\rho = N_{12}^+ + 2(d-1)N_3 \qquad J = N_{12}^- \qquad \rho_J = \rho_1 - 2(d-1)N_3 \qquad \kappa_J = \kappa_{12}^- \\ &L_\rho = \partial_t\rho + \partial_x J = 0 \qquad j_1 = \xi\rho_1 \qquad S_\rho = S_{12}^+ + 2(d-1)S_3 = 0 \\ &L_J = \partial_t J + \partial_x \rho_J = \partial_z(-\xi J + \rho_J) = (\alpha\rho + \eta)(\rho\kappa_J - J) + S_J \qquad S_J = S_{12}^- \\ &L_3 = \partial_t N_3 = -\xi\partial_z N_3 = R_3 = \sigma(N_1N_2 - N_3^2)/(d-1) + (\alpha\rho + \eta)(\kappa_3\rho - N_3) + S_3 \end{aligned} \tag{5.1}$$

$$N_i = n_{0i} + n_i N(z) \qquad \rho = \rho_0 + \rho_1 N(z) \qquad J = j_0 + j_1 N(z). \tag{5.2}$$

5.1. *Conservative Broadwell model:* $N_i(z = x - \xi t)$, $i = 1, 2, 3$ *and* $S_i = \alpha = \eta = 0$

$$(1-\xi)N_{1,z} = -(1+\xi)N_{2,z} = (d-1)\xi N_{3,z} = \sigma(N_3^2 - N_1N_2). \tag{5.3}$$

The interest (appendix A1), for this DVM which has been the most extensively studied, is an analytical proof (with V_0 giving the shock direction and shock inequalities satisfied) that the only possible shocks are compressive, contrary to the nonconservative case where they are only rarefactive. First, we show (lemma 2) that we know the states at $\pm\infty$ from the sign of $\xi\rho_1$ and (lemma 4) that the sign of V_0 is known from $\xi V_0 < 0$. Second, we show (lemma 3), for the $d = 2$ model, that the shock speed satisfies $|\xi| < 1$ (and for the isotropic (0) state for $d = 3$). Third (theorem 3), we show that the shocks are compressive. The main difference between the two theories, for the same model, is that the conservative sound waves are roots of a quadratic polynomial while the nonconservative (except mass) sound waves are roots of a linear polynomial. For the conservative Broadwell model, with two conservation relations, the last evolution equation is a scalar Riccati equation which is completely integrable [9]. This means that the presented results are general and not restricted to a particular class of solutions.

5.2. Nonconservative Broadwell model: $S_i \neq 0, \alpha \neq 0$

Linearizing around (0) we get a 3×3 X_ρ, X_J, X_3 linear system with three operators: $\Omega_3 = \partial_t(\partial^2_{t^2} - \partial^2_{x^2})$, (roots $\pm 1, 0$), Ω_2, $\Omega_1 = A_J B_0(\partial_t + \xi_0 \partial_x) \to \xi_0$ written in (3.1)–(3.3):

$$\Omega_3 + \Omega_2 + \Omega_1 = \begin{vmatrix} \partial_t & \partial_x & 0 \\ \partial_x - A_\rho & \partial_t + A_J & -2(d-1)\partial_x \\ B_\rho & B_J & \partial_t + B_0 \end{vmatrix} = 0$$

$$B_0 := A_J + \sigma(n_{01} + n_{02} + 2n_{03}/(d-1)) > 0 \qquad B_J 2(d-1) = \sigma j_0 > 0 \tag{5.4}$$

$$B_\rho = -(2\alpha\rho_0 + \eta)\kappa_3 + \alpha n_{03} - \sigma(n_{01} + n_{02})/2$$

$$\Omega_2 = \partial^2_{t^2}(B_0 + A_J) + \partial^2_{xt}(2(d-1)B_J + A_\rho) - \partial^2_{x^2}(B_0 + 2(d-1)B_\rho).$$

With $A_J B_0 > 0$, $B_0 + A_J > 0$ part of the Whitham conditions are satisfied.

For Ω_2 we associate $P_2(\zeta)$ with roots $P_2(\zeta_{0\pm}) = 0$. From positivity we get $P_2(\pm 1) > 0$ for $d = 2, 3$ while only for $d = 2$ and symmetric models $\kappa_i = \frac{1}{4}$ we get analytically $P_2(0) = -[\eta/2 + 2(\sigma + \alpha)n_{03}] < 0$ with the interlacing Whitham property $-1 < \zeta_- < 0 < \zeta_+ < 1$ satisfied for Ω_3, Ω_2 (in the other cases the property has been only verified numerically).

For Ω_2, Ω_1, the Whitham criteria $\zeta_{0-} < \xi_0 < \zeta_{0+}$ and $\zeta_{s-} < \xi_s < \zeta_{s+}$, have only been verified numerically.

The exact solutions (5.2) and section 4 studied in appendix A2, result from the compatibility between two scalar Riccati equations for J, N_3 $(\gamma_J = \gamma_3)$. The mass conservation gives the jump relation between the asymptotic states while both the nonconservative momentum and N_3 evolution equations give four relations for the asymptotic states and two for the compatible Riccati solutions.

We recall [10] that from the knowledge of the (0), (s) states we can predict the existence (or not) of overshoots for the internal energy $\simeq P/\rho$, depending whether $j_0 j_s < 0$ (or not). (For Broadwell and hexagonal models we have $P \simeq \rho[1 - (j/\rho)^2]$ for the pressure.)

In the figures 2(a)–(b), for $d = 2, 3$, see table 2 (with Lax–Whitham criteria and shock inequalities satisfied) we have rarefactive shocks. ρ_0, ξ_0, P_0 are at $z = +\infty$, but, except for figure 2(a), $d = 2$ with upstream at $z = +\infty$, in all other cases they are at $-\infty$. In figure 2(a) (figure 2(b)) we have P/ρ with (without) overshoots.

In the original Broadwell model the loss and gain elastic collision terms are zero at one asymptotic state. Here (except in figure 2(a), $d = 2$) we choose such solutions at the (0) state with $n_{01} = 1$, $n_{02} = n_{03} = 0$. Consequently, $N_i(z)/n_{si}$ are equal for $i = 2,\ 3$, $\rho_0 = j_0 = 1$ and for the pressure $P_0 = 0$. We notice that we can have either an overshoot in figure 2(a), $d = 3$ or monotonic internal energy in figure 2(b). In contrast for 'homogeneous' solutions (all densities equal at one asymptotic state), due to either $j_0 = 0$ or $j_s = 0$, the product $j_0 j_s = 0$ and we cannot have overshoots.

We notice that for the solutions presented in figures 2(a) and (b) (as in figures 3(a) and (b), 4(a) and (b)) the S_i do not have the same sign, they can be either sources or sinks. The reason is that in order to retain the mass conservation the global amount of sources and sinks must be zero, contrary to the pure nonconservative models where we can have either only sources or sinks.

Table 1. Broadwell models.

Figure	Dimension	$\xi_0 < \xi < \xi_s$	$\zeta_{0\pm}$	$\zeta_{s\pm}$	$W_0 > V_0 > V_s > W_s$	$\rho_0\ \rho_s$	$P_0\ P_s$					
2(a)	2	0.17, 0.4, 0.44	0.79, −0.63	0.73, −0.91	−[0.0007, 0.24, 0.68, 0.71]	2.2 > 0.76	2.1 > 0.71					
2(a)	3	−0.37, −0.27, −0.24	0.54, −0.69	0.55, −0.69	1.37, 1.27, 0.057, 0.028	1.0 < 2.2	0.0 < 2.1					
2(b)	2	0.048, 0.12, 0.14	0.66, −0.48	0.69, −0.52	0.9, 0.88, 0.38, 0.36	1.0 < 2.3	0.0 < 1.72					
2(b)	3	0.16, 0.166, 0.17	0.76, −0.28	0.75, −0.32	0.84, 0.83, 0.41, 0.4	1.0 < 2.0	0.0 < 1.35					
		α	η	$j_0\, j_s$	γ	$\xi_\infty < \xi_{-\infty}$	κ_1	κ_2	κ_3	S_1	S_2	S_3
2(a)	2	10^{-2}	3×10^{-4}	−	+	$\xi_0 < \xi_s$	0.51	0.28	0.1	−0.864	−0.871	0.868
2(a)	3	0.5	3.9	−	+	$\xi_0 < \xi_s$	0.28	0.51	0.0525	3.2	−2.3	−0.23
2(b)	2	10^{-2}	4.9×10^{-3}	+	+	$\xi_0 < \xi_s$	0.51	0.28	0.105	0.028	−0.016	−0.006
2(b)	3	0.5	5×10^{-3}	+	+	$\xi_0 < \xi_s$	0.51	0.28	0.0525	2.7	−1.54	−0.29

Table 2. Hexagonal model.

Figure	$\xi_s < \xi < \xi_0$	$\zeta_{s\mp}$	$\zeta_{0\mp}$	$W_0 < V_0$	$V_s < W_s$	$\rho_s < \rho_0$	$P_s < P_0$						
3(*a*)	−0.55, −0.26, −0.19	−0.71, 0.36	−0.49, 0.35	0.05, 0.12	0.53, 0.82	0.45 < 1.95	0.42 < 1.91						
3(*b*)	−0.038, −0.22, −0.13	−0.62, 0.39	−0.46, 0.38	0.26, 0.35	0.61, 0.76	1.6 < 2.7	1.34 < 2.6						
	α	η	$j_0 j_s$	γ	$\xi_\infty < \xi_{-\infty}$	κ_1	κ_2	κ_3	κ_4	S_1	S_2	S_3	S_4
3(*a*)	5.2	0.33	−	−	$\xi_s < \xi_0$	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{3}{12}$	0.09	0.128	−0.04	−0.26
3(*b*)	6.61	0.204	+	−	$\xi_s < \xi_0$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	1.56	1.3	−0.63	−2.91

5.3. Comparison between conservative and nonconservative Broadwell models

The main difference is that the shocks are compressive (rarefactive) for conservative (nonconservative) models. In the conservative case the two main properties are $\gamma\xi\rho_1 > 0, \xi V_0 < 0$ (lemmas 2–4), while in the nonconservative case they are $\gamma(\kappa_J - \xi)\rho_1 > 0, (\kappa_j - \xi)V_0 > 0$ (theorem 2, lemma 1(i)). In the four cases (theorems 1–3 with (1.1)–(2.2)), the locations of the (0) or (s) states at $\pm\infty$ as well as $\rho_s \gtrless \rho_0$ are the same. The difference is that the directions of the shocks (sign of V_0) are opposite, such that from the upstream to the downstream states, ρ increases (conservative) or decreases (nonconservative).

6. Hexagonal $6v_i$ model with $\rho = N^+_{1,4} + 2N^+_{2,3}$, $J = N^-_{1,4} + N^-_{2,3}$

We have four independent densities (cf figure 1) N_1, N_4 and N_2, N_3 (multiplicity two) with x-coordinates $(\pm 1, \pm\frac{1}{2})$ and the four evolution equations written in appendix B1. First, for evolution equations without elastic collisions, we still have both the mass conservation (2.4*b*)–(2.5*b*), with the jump relation $j_1 = \rho_1\xi$ and $S_\rho = S^+_{14} + 2S^+_{23} = 0$, and the modified momentum (2.4*c*)–(2.5*c*) with: $\kappa_J = \kappa^+_{12} - \kappa^+_{34}$, $\rho_J = \rho - 3N^+_{23}/2$. Second, with only binary elastic collisions we have another relation without elastic collisions for $N^-_{2,3}$.

In the conservative model this is a 'spurious conservation law' with sound-wave roots of a cubic polynomial (cf the study of shock waves in [6]), *while for the true physical hexagonal model, adding cubic elastic collisions, the sound waves are roots of a quadratic polynomial*. Here for the nonconservative (except mass) model the 'spurious relation' disappears and we can restrict our study to binary collisions, but for this reason we will not compare the conservative and nonconservative shock solutions.

To the mass conservative and momentum nonconservative relations we add a third $N_2 - N_3$ evolution equation without elastic collisions (it is a 'spurious relation' for the conservative model) and finally the N_2 equation with elastic collisions:

$$\begin{aligned} &\partial_t N^-_{2,3} + \partial_x N^+_{2,3}/2 = \partial_z(-\xi N^-_{2,3} - N^+_{23}) = (\alpha\rho + \eta)(\rho\kappa^-_{2,3} - N^-_{2,3}) + S^-_{23} \\ &(\partial_t + \partial_x/2)N_2 = \partial_z(-\xi + 1/2)N_2 = \sigma(N_1N_4 - N_2N_3) + (\alpha\rho + \eta)(\rho\kappa_2 - N_2) + S_2. \end{aligned} \tag{6.1}$$

The exact solutions (4.1)–(4.3), studied in appendix B1, result from the compatibility between three scalar Riccati equations for J, $N^-_{2,3}$, N_2 requiring $\gamma_J = \gamma_{23} = \gamma_2$.

Linearizing around the (0) state we have four operators: $\Omega_4 = (\partial^2_{t^2} - \partial^2_{x^2})(\partial^2_{t^2} - \partial^2_{x^2}/4)$. (roots $\pm 1, \pm\frac{1}{2}$), Ω_3, Ω_2 (written in appendix B2) (roots $\zeta_{0\pm}$ with *verification* $\xi_0 \in [\zeta_{0+}, \zeta_{0-}]$ *numerical*) and Ω_1 for the sound waves written in (3.1)–(3.3).

We present, see table 2, in figures 3(*a*) and (*b*) solutions with the Lax–Whitham criteria and shock inequalities satisfied, and rarefactive shocks. In figures 3(*a*) and (*b*) we have downstream ρ_s, ξ_s, P_s at $z = +\infty$, and upstream (0) state at $z = -\infty$ and $j_0 j_s \lessgtr 0$ with (without) P/ρ overshoot.

7. Squares $8v_i$ ($9v_i$) without (with) rest particle

7.1. Square $8v_i$ model with $\rho = N^+_{1,5} + 2(N^+_{2,4} + N_3)$, $J = N^-_{1,5} + 2N^-_{2,4}$

We have five independent densities (cf figure 1) N_1, N_5 and N_2, N_3, N_4 (multiplicity two) with x-coordinates $(\pm 1, 1, 0, -1)$ and the five evolution equations written in appendix C1. We have three equations without elastic collisions. First, the mass conservation (2.4*b*)–(2.5*b*) still with the jump relation $j_1 = \rho_1\xi$ and $S_\rho = S^+_{15} + 2(S^+_{24} + S_3) = 0$, second the nonconservative momentum (2.4*c*)–(2.5*c*) with $\kappa_J = \kappa^-_{15} + 2\kappa^-_{24}$, $\rho_J = \rho - 2N_3$ and third the nonconservative

Table 3. Square $8v_i$ model.

$\alpha, \eta, \rho_s < \rho_0$	$\kappa_1, \kappa_2, \kappa_3, \kappa_4, \kappa_5$	$S_1, S_2, S_3, S_4, S_5, P_s < P_0$
0.1, 0.31, 0.92 < 0.93	3.10^{-3}, 0.296, 5.10^{-3}, 0.013	−0.24, 0.37, −0.277, 0.3, 0.04, 0.1075, 0.693 < 0.73
$\xi_0 < \xi < \xi_s$	$\zeta_{0+} < \zeta_{0-}, \zeta_{s-} < \zeta_{s+}$	$W_s < V_s < V_0 < W_0$
0.849, 0.85, 0.855	−0.639, 0.908, −0.638, 0.914	−1.528, −1.522, −1.506, −1.501

relation for the energy or equivalently for $N_{24}^+ = N_2 + N_4$. Finally, we write the equations for N_2, N_3 with two elastic collision terms C_1, C_2:

$$\begin{aligned} &\partial_t N_{2,4}^+ + \partial_x N_{2,4}^- = \partial_z(-\xi N_{2,4}^+ + N_{2,4}^-) = (\alpha\rho + \eta)(\rho\kappa_{2,4}^+ - N_{2,4}^+) + S_{2,4}^+ \\ &(\partial_t + \partial_x)N_2 = \partial_z(-\xi + 1)N_2 = \sigma_2 C_2 + (\alpha\rho + \eta)(\rho\kappa_2 - N_2) + S_2 \qquad C_2 := N_1N_4 - N_2N_5 \\ &\partial_t N_3 = -\xi\partial_z N_3 = \sigma_1 C_1 + (\alpha\rho + \eta)(\rho\kappa_3 - N_3) + S_3 \qquad C_1 := N_1N_5 - N_3^2. \end{aligned} \tag{7.1}$$

The exact solutions (4.1)–(4.3), studied in appendix C1, result from the compatibility between the four scalar Riccati equations for $J, N_{2,4}^+, N_2, N_3$ with $\gamma_J = \gamma_{2,4} = \gamma_3 = \gamma_2$.

Linearizing around the (0) *state* we have five operators: $\Omega_5 = \partial_t(\partial_{t^2}^2 - \partial_{x^2}^2)^2$ (roots $0, \pm 1$), Ω_4, Ω_3 and Ω_2 with roots $\zeta_{0,\pm}$ (written in appendix C2) and only a *numerical verification* of the Whitham criterion $\xi_0 \in [\zeta_{0+}, \zeta_{0-}]$ with ξ_0 for Ω_1 (see (3.1)–(3.3)). In figure 4(*a*), see table 3, we present a rarefactive shock with the Lax–Whitham and shock inequalities satisfied and $(\rho_s, P_s) < (\rho_0, P_0)$, (0) being the upstream at ∞ and (s) the downstream at $-\infty$.

7.2. Square $9v_i$ model with $\rho = N_0 + N_{1,5}^+ + 2(N_{2,4}^+ + N_3)$, $J = N_{1,5}^- + 2N_{2,4}^-$

To the previous $N_i, i = 1 \ldots 5$ we add a rest particle N_0 with x-coordinate (0). In the mass conservation (2.4*b*)–(2.5*b*), with $j_1 = \rho_1\xi$, we have $S_\rho = S_0 + S_{1,5}^+ + 2(S_{2,4}^+ + S_3) = 0$ and in the nonconservative momentum relation: $\kappa_J = \kappa_{1,5}^- + 2\kappa_{2,4}^-$, $\rho_J = \rho - 2N_3 - N_0$.

We have a third equation, without elastic collisions, for the energy or equivalently for $N_{2,4}^+ - N_0/2 := N_{240}$ with $\kappa_{240} = \kappa_{2,4}^+ - \kappa_0/2$, $S_{240} = S_{24}^+ - S_0/2$. We add three N_{23}^+, N_0, N_{34}^+ equations with four elastic collision terms $C_1, C_2, C_0, \bar{C}_0$ that we write:

$$\begin{aligned} &\partial_t z(-\xi N_{240} + N_{2,4}^-) = (\alpha\rho + \eta)(\rho\kappa_{240} - N_{240}) + S_{240} \qquad C_0 = N_1N_3 - N_0N_2 \\ &-\partial_z\xi N_0 = 2\sigma_0[C_0 + \bar{C}_0] + (\alpha\rho + \eta)(\rho\kappa_0 - N_0) + S_0 \qquad \bar{C}_0 = N_5N_3 - N_0N_4 \end{aligned} \tag{7.2}$$

$$\begin{aligned} &\partial_z(-\xi N_{2,3}^+ + N_2) = \sigma_2 C_2 + \sigma_1 C_1 - \sigma_0\bar{C}_0 + (\alpha\rho + \eta)(\rho\kappa_{23}^+ - N_{2,3}^+) + S_{2,3}^+ \\ &\partial_z(-\xi N_{3,4}^+ - N_4) = -\sigma_2 C_2 + \sigma_1 C_1 - \sigma_0 C_0 + (\alpha\rho + \eta)(\rho\kappa_{3,4}^+ - N_{3,4}^+) + S_{3,4}^+. \end{aligned} \tag{7.3}$$

The exact solutions (4.1)–(4.3), studied in appendix D, result from the compatibility between five scalar Riccati equations for $J, N_{240}, N_{2,3}^+, N_0, N_{3,4}^+$ with $\gamma_J = \gamma_{240} = \gamma_{23} = \gamma_0 = \gamma_{34}$.

Linearizing around the (0) state we have six operators: $\Omega_6 = \partial_t^2(\partial_{t^2}^2 - \partial_{x^2}^2)^2$ (roots $0, \pm 1$), $\Omega_5, \Omega_4, \Omega_3, \Omega_2$ (not presented) and Ω_1, with ξ_0 written in (3.1)–(3.3).

In figure 4(*b*), see table 4, we present a solution ($\bar{S}_i := 10^6 S_i, \bar{\kappa}_i := 10^3\kappa_i, \bar{\rho}_0 := 10^3\rho_0, \bar{P}_0 := 10^3 P_0, \ldots$), with the Lax–Whitham and shock inequalities satisfied, which is a rarefactive shock. The upstream (s) state is at $-\infty$ and the (0) downstream at ∞ with $(\rho_s, P_s) > (\rho_0, P_0)$.

Table 4. Square $9v_i$ model.

α, η	$\bar{\kappa}_0, \bar{\kappa}_1, \bar{\kappa}_2, \bar{\kappa}_3, \bar{\kappa}_4, \bar{\kappa}_5$	$\bar{S}_0, \bar{S}_1, \bar{S}_2, \bar{S}_3, \bar{S}_4, \bar{S}_5$
1.0, 0.02	58.0, 45.0, 40.0, 54.0, 233.0, 239.0	4.0, 25.0, 16.0, 17.0, −35.0, −27.0
$\xi_0 < \xi < \xi_s$	$\bar{\rho}_0 < \bar{\rho}_s$, $\bar{P}_0 < \bar{P}_s$	$W_s < V_s < V_0 < W_0$
−0.76, −0.7, −0.66	9.0, 23.0, 12.0, 29.0	0.235, 0.274, 0.7, 0.757

8. Conclusion

Our motivation was to find an intermediate class of models, between the standard nonconservative models and the conservative ones, such that some very useful properties of the conservative models could subsist. This is possible with a restriction of the nonconservative parameters such that only the mass conservation law (with macroscopic quantities mass and momentum) be retained. For these models the amount of creation–annihilation particles and sources and sinks is globally zero for the mass conservation, but remains at the microscopic level for the densities, as well as for the other conservation laws which do not only share macroscopic quantities. Since the pioneering Broadwell work, contrary to the nonconservative DVMs, a major application of the conservative DVMs has been the study of shock waves. For the modified nonconservative models (with mass conservation), for travelling waves, we discuss both the Lax–Whitham criteria and shock inequalities. For these models (more than $2v_i$), with momentum conserved in the associated conservative DVMs, we have found a *general result from only the conservative mass and nonconservative momentum evolution equations*. The sound waves are the same for all models and depend on the nonconservative parameters. Consequently, even without explicit solutions, we can discuss the Lax criterion and shock inequalities (section 3). With the shock direction given by the sign of the four quantities $V_{\pm\infty} = V_{\pm}$, $W_{\pm\infty} = W_{\pm}$ (the ratio of the momentum to the mass minus either the shock speed ξ or the sound-wave speeds $\xi_{\pm\infty}$) and the shock inequalities satisfied we deduce the Lax criterion $\xi_{+\infty} < \xi < \xi_{-\infty}$.

We show that only rarefactive shocks exist, in contrast with conservative models where, in general, the shocks are compressive while rarefactive shocks have been found. As an illustration, for the *conservative Broadwell models we show that only compressive shocks exist*.

In sections 4–6 we construct explicitly compatible scalar Riccati solutions (detailed work is found in the appendices) and verify the above general results. The new result is the determination of a parameter C_{Lax} such that either $C_{Lax} > 0$ (or < 0) leads to both the Lax criterion and satisfied (or not) shock inequalities. For the Whitham interlacing criterion we do not have general analytic results and the verification is numerical.

Counting argument. Adding one more independent density N_i we only have three more relations but four parameters $\kappa_i, S_i, n_{0i}, n_i$ and we can hope (if positivity is satisfied) to obtain new compatible Riccati solutions for more general models. However, I think it more useful to consider these compatible Riccati solutions *satisfying boundary conditions* as has been done many times for the conservative models. Another possible application of these solutions for the presented nonconservative models (done recently in discrete conservative models [11]) is the study of evaporation–condensation processes with inversion of the internal energy. For conservative models nonmonotonic internal energy behaviour can be predicted from the macroscopic quantities (here we have also verified that this occurs for the presented nonconservative models) and the same prediction was found for internal energy inversion [11].

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Appendix A. Shock waves for the Broadwell models

Appendix A.1. Conservative Broadwell models written in (5.3)

Appendix A.1.1. Solutions for $d = 2, 3$ and $|\xi| < 1$. All $N_{i,z}$ being proportional we can write for $N_i(z)$, $\rho(z) = N^+_{1,2} + 2(d-1)N_3$, $J(z) = N^-_{1,2}$ the (3.2) travelling waves solutions with two asymptotic states: (0) n_{0i}, ρ_0, j_0 and (s) $n_{si} = n_{0i} + n_i$, $\rho_s = \rho_0 + \rho_1$, $j_s = j_0 + j_1$ such that the associated collision terms are zero $N_1N_2 - N_3^2 = 0$ for $N_i = n_{0i}, n_{si}$:

$$n_{01}n_{02} = n_{03}^2 \qquad n_1n_2 - n_3^2 + n_1n_{02} + n_2n_{01} - 2n_3n_{03} = 0. \tag{A.1}$$

We define $\tilde\sigma = \sigma/(d-1)$, substitute $N_i(z) = n_{0i} + n_iN(z)$ into (5.3), for $N_{3,z}$ get a Riccati equation [9] for N that we integrate:

$$\xi n_3N_z = \tilde\sigma(n_1n_2 - n_3^2)N(N-1) \qquad N(z) = [1 + \bar d \mathrm{e}^{\gamma z}]^{-1} \qquad \gamma = \tilde\sigma(n_3^2 - n_1n_2)/n_3\xi \tag{A.2}$$

with $\rho_\infty = \rho_0, \rho_s$ if $\gamma \gtrless 0$. From ρ_0, ρ_s, ξ we want to know the states at $\pm\infty$ (given by the γ sign). We define $\bar n_i = n_i/\rho_1$ and from the mass and momentum conservation laws get $\bar n_i$ as a function of ξ, substitute into (A.2) and get γ:

$$\begin{aligned} &j_1/\rho_1 = \xi = \bar n^-_{1,2} \qquad \bar n^+_{1,2} = \xi\bar n^-_{1,2} = \xi^2 \qquad 1 = \bar n^+_{1,2} + \bar n_2 + 2(d-1)\bar n_3 \\ &\bar n_3 = (1-\xi^2)/2(d-1) \qquad \bar n_1 = \xi(\xi+1)/2 \qquad \bar n_2 = \xi(\xi-1)/2 \\ &\gamma\xi/\rho_1 = \tilde\sigma[\xi^2((d-1)^2 - 1) + 1]/2(d-1) > 0 \rightarrow \gamma\xi\rho_1 > 0. \end{aligned} \tag{A.3}$$

Lemma 2. *If $\rho_1\xi \gtrless 0$ then $\rho_{+\infty} = \rho_0, (\rho_s)$, $\rho_{-\infty} = \rho_s(\rho_0)$.*

For the proof we get $\gamma \gtrless 0$ from (A.3) and the result from (A.2).

Lemma 3. *The shock speed satisfies, from positivity, $|\xi| < 1$.*

From (A.1)–(A.3) we get (for brevity we prove this only for $d = 2$) two quadratic ξ polynomials:

$$P_2(\xi) = \xi^2(\rho_0 + \rho_s) - 2\xi j_s + \rho_1 - 4n_{s3} \qquad \tilde P_2(\xi) = \xi^2(\rho_0 + \rho_s) - 2\xi j_0 - \rho_1 - 4n_{03} \tag{A.4}$$

deduce $P_2(1) = 4n_{s2} > 0$, $P_2(-1) = 4n_{s1} > 0$, $P_2(0) = \rho_1 - 4n_{03} < 0$ if $\rho_1 < 0$ and $\tilde P_2(1) = 4n_{02} > 0$, $\tilde P_2(-1) = 4n_{01} > 0$, $\tilde P_2(0) = -\rho_1 - 4n_{03} < 0$ if $\rho_1 > 0$. In both cases $\rho_1 \gtrless 0$ we deduce $|\xi| < 1$ and for the two roots $\xi^\pm \gtrless 0$. For the $d = 3$ model, for brevity we consider only the isotropic case: $n_0 = n_{0i}$, $i = 1, 2, 3$, $j_0 = 0$.

Lemma 3(i). *For the isotropic* (0) *state, the shock speed satisfies $1/d < \xi^2 < 1$ for $\rho_1 > 0$ and $\xi^2 < 1/d$ for $\rho_1 < 0$.*

For the proof we rewrite (A.1)–(A.3), get $\xi^2 = \rho_s/(\rho_0 + \rho_s)$ for $d = 2$ and explicitly for $d = 3, 2$:

$$d = 2, 3: \quad 0 < n_0 = \rho_1(1-\xi^2)[1 + \xi^2((d-1)^2 - 1)]/4(d-1)(d\xi^2 - 1). \tag{A.5}$$

Appendix A.1.2. Linearizing (A.1) with $N_i \simeq n_{0i}[1 + X_i(x - \zeta t)]$ we get a linear X_i system leading to a cubic polynomial $P_3^{(0)}$ $(\zeta = 0, \pm 1)$ and a quadratic $P_2^{(0)}(\zeta = \xi_0^{\pm}) = 0$ for the sound waves.

$$P_2^{(0)}(\xi_0) = \alpha_0 \xi_0^2 - \xi_0(d-1) j_0 - 2n_{03} \qquad \alpha_0 := 2n_{03} + (d-1)(n_{01} + n_{02}) > 0 \qquad \xi_0^+ \xi_0^- < 0. \tag{A.6}$$

With $P_2^{(0)}(0) < 0$, $P_2^{(0)}(\pm 1) > 0$ the Whitham criterion for $P_3^{(0)}$, $P_2^{(0)}$ is satisfied, $-1 < \xi_0^- < 0 < \xi_0^+ < 1$ and similarly for $P_2^{(s)}(\xi_s^{\pm})$ with $n_{0i} \to n_{si}$.

Lemma 4. *We find* $\xi V_0 < 0$ *for* $d = 2, 3$.

For the proof we *begin with* $d = 2$. First we study the solutions $P_2^{(0)}(\xi_0^{\pm}) = 0$ of (A.6):

$$\xi_0^{\pm} = (j_0 \pm \sqrt{\Delta})/2\rho_0) \gtrless 0 \qquad \Delta = j_0^2 + 8n_{03}\rho_0 \qquad W_0^{\pm} = j_0/\rho_0 - \xi_0^{\pm} = \xi_0^{\mp} \lessgtr 0 \tag{A.7}$$

and similarly $\xi_s^{\pm} \gtrless 0$, $W_s^{\pm} = j_s/\rho_s - \xi_s^{\pm} = \xi_s^{\mp} \lessgtr 0$. Second, with the sign $W_0^{\pm} = \xi_0^{\mp}$, $W_s^{\pm} = \xi_s^{\mp}$, we have the same index $\xi_0^{\pm}$, $\xi_s^{\pm}$ for ξ_0, ξ_s. From lemma 1 ξ is in the interval $\xi_0^{\pm}$, $\xi_s^{\pm}$ and $\xi W_0^{\pm} = \xi \xi^{\mp} < 0$ or $\xi V_0 < 0$. In conclusion we must only associate:

$$\xi \gtrless 0 \to \xi_0 = \xi_0^{\pm} \qquad \xi_s = \xi_s^{\pm} \qquad W_0 = W_0^{\pm} = \xi_0^{\mp} \lessgtr 0 \qquad W_s = W_s^{\pm} = \xi_s^{\mp} \lessgtr 0.$$

We continue with $d = 3$ and notice that for $d = 2$ the necessary property was $W_0^{\pm} \lessgtr 0$, $W_s^{\pm} \lessgtr 0$ associated to $\xi_0^{\pm}$, $\xi_s^{\pm}$. In order to show the same property we only rewrite $\xi_0^{\pm}$, $W_0^{\pm}$:

$$\begin{aligned} &\xi_0^{\pm} = [j_0 \pm \sqrt{\Delta}]/2\Sigma \gtrless 0, \quad \Sigma = \Sigma_{i=1}^3 n_{0i} > 0 \qquad \Delta = j_0^2 + 4n_{03}\Sigma \\ &2\Sigma\rho_0 W_0^{\pm} = (n_{01} + n_{02})(j_0 \mp \sqrt{\Delta}) \mp 2n_{03}(2\sqrt{\Delta} \pm j_0) \lessgtr 0 \to W_s^{\pm} \lessgtr 0. \end{aligned} \tag{A.8}$$

Let us notice that for isotropic (0) state $n_{0i} = n_0$, from (A.6), (A.7) we get $\xi_0^{\pm} = 1/\sqrt{d}$ and recall, from lemma 3(i), $|\xi| \gtrless 1/\sqrt{d}$ if $\rho_1 \gtrless 0$.

Theorem 3. *For* $d = 2, 3$ *no rarefactive shock can exist.*

For the proof we must show that ρ at the upstream is lower than at the downstream (or compressive shock). We apply (A.3) and lemma 4: $\gamma\rho_1\xi > 0$, $\xi V_0 < 0$ to four cases.

(1) $\xi > 0$ giving $V_0 < 0$, a negative direction of the shock with upstream (downstream) at $\pm\infty$.

- (1.1) $\rho_1 > 0$ giving $\gamma > 0$ and $\rho_0(\rho_s)$ at $\pm\infty$ or ρ_0 (upstream) $< \rho_s$ (downstream).
- (1.2) $\rho_1 < 0 \to \gamma < 0$ and $\rho_s(\rho_0)$, upstream (downstream) at $\pm\infty$or ρ_s (upstream) $< r_0$ (downstream).

(2) $\xi < 0$ giving $V_0 > 0$, a positive direction of the shock, upstream (downstream) at $\mp\infty$.

- (2.1) $\rho_1 > 0 \to \gamma < 0$ and $\rho_s(\rho_0)$ at $\pm\infty$ or downstream (upstream) and finally ρ_0 (upstream) $< \rho_s$ downstream.
- (2.2) $\rho_1 < 0 \to \gamma > 0$ and $\rho_s(\rho_0)$, upstream (downstream) at $\mp\infty$ or ρ_s (upstream) $< \rho_0$ (downstream).

Appendix A.2. Nonconservative Broadwell model with Riccati solutions (5.2)

We define:

$$\bar{n}_i = n_i/\rho_1 \quad \bar{\eta} = \eta/\alpha \quad \bar{\gamma} = \gamma/\rho_1\alpha \quad \bar{n}_{ijkl} = \bar{n}_i\bar{n}_j - \bar{n}_k\bar{n}_l \quad \bar{\sigma} = \sigma/(d-1)\alpha$$
$$\bar{n}_{0,ijkl} = \rho_1\bar{n}_{ijkl} + n_{0i}\bar{n}_j + n_{0j}\bar{n}_i - n_{0k}\bar{n}_l - n_{0l}\bar{n}_k \qquad C_{Lax} = 1 - \xi^2 - 2(d-1)\bar{n}_3. \tag{A.9}$$

From the mass conservation (2.4b) we get $S_\rho = S^+_{1,2} + 2(d-1)S_3 = 0$ and one relation $j_1 = \rho_1\xi$. With $\rho_1 = n^+_{1,2} + 2(d-1)n_3$ we deduce $\bar{n}_i$ as functions of $\bar{n}_3$:

$$1 = \bar{n}^+_{1,2} + 2(d-1)\bar{n}_3 \qquad \xi = \bar{n}^-_{1,2} \to 2\bar{n}_i = 1 - (-1)^i\xi - 2(d-1)\bar{n}_3 \qquad i = 1, 2. \tag{A.10}$$

For both the momentum relation without elastic collisions (2.4c)–(2.5c) and the N_3 evolution equation with elastic collisions (2.4a)–(2.5a) we get three relations: two for the (0), (s) asymptotic states and one for the scalar Riccati solutions:

$$S_J = S^-_{12} = \alpha(\rho_0 + \bar{\eta})(j_0 - \kappa_J\rho_0) \qquad \text{and} \qquad \rho_0, j_0 \to \rho_s, j_s$$
$$A_\rho := \rho_s + \bar{\eta} \qquad B_\rho := A_\rho + \rho_0 \qquad \xi A_\rho = \kappa_J B_\rho - j_0 \qquad \bar{\gamma} = (\kappa_J - \xi)/C_{Lax} \tag{A.11}$$

$$S_3 = \bar{\sigma}(n^2_{03} - n_{01}n_{02}) + \alpha(n_{03} - \kappa_3\rho_0)(\rho_0 + \bar{\eta}) \qquad \text{and} \qquad n_{0i}, \rho_0 \to n_{si}, \rho_s$$
$$B_\rho\kappa_3 = A_\rho\bar{n}_3 + n_{03} - \bar{\sigma}\bar{n}_{0,1233} \qquad \bar{\gamma} = (\bar{n}_3 - \kappa_3 - \bar{\sigma}\bar{n}_{1233})/\xi\bar{n}_3. \tag{A.12}$$

Starting with $\rho_1, \xi, n_3, \kappa_J, \alpha, \eta, n_{01}, n_{02}$ we deduce successively all other parameters.

Appendix B. Shock waves for the hexagonal model (4.1)

Appendix B.1.

Compatible Riccati solutions for the four N_i evolution equations

$$\partial_t N_i + e_i\partial_x N_i = \sigma_i(N_2N_3 - N_1N_4) + (\alpha\rho + \beta)(\rho\kappa_i - N_i) + S_i \qquad i = 1, 2, 3, 4 \tag{B.1}$$

$e_i = 1, \pm\frac{1}{2}, -1$, $\sigma_i/\sigma = 2, 1, 1, 2$. As in (A.9) we define $\bar{n}_i, \ldots$ and $n^\pm_{0,ij} = n_{0i} \pm n_{0j} \ldots$.:

$$\bar{\sigma}_i = \sigma_i/\alpha \quad C_{Lax} = 1 - \xi^2 - 3\bar{n}^+_{2,3}/2 \quad (\rho_1, j_1) \to 1 = \bar{n}^+_{1,4} + 2\bar{n}^+_{2,3} \quad \xi = \bar{n}^-_{1,4} + \bar{n}^-_{2,3}. \tag{B.2}$$

We retain three evolution equations without elastic collisions for $\rho(z)$, $J(z)$, $N^-_{23}(z)$ and $N_2(z)$ with elastic collisions. For ρ the relation $j_1 = \xi\rho_1$ is written at the end of (B.2), for each of the three other we have three relations, two for the asymptotic states and one $\bar{\gamma}$ for the compatible Riccati solutions:

$$\bar{\gamma} = (\kappa_J - \xi)/C_{Lax} = (\kappa^-_{2,3} - \bar{n}^-_{2,3})/(\bar{n}^+_{2,3} - \xi\bar{n}^-_{2,3}) = [\bar{\sigma}\ \bar{n}_{1423} + \kappa_2 - \bar{n}_2]/\bar{n}_2(\tfrac{1}{2} - \xi) \tag{B.3}$$

$$B_\rho\kappa_J = A_\rho\xi + j_0 \qquad B_\rho\kappa^-_{2,3} = A_\rho\bar{n}^-_{2,3} + n^-_{0,23} \qquad B_\rho\kappa_2 = A_\rho\bar{n}_2 + n_{02} - \bar{\sigma}\bar{n}_{0,1423} \tag{B.4}$$

$$S_J = S^+_{12} - S^+_{34} = \alpha(\rho_0 + \bar{\eta})(j_0 - \rho_0\kappa_J) \qquad S^-_{23} = \alpha(n^-_{0,23} - \rho_0\kappa^-_{23})$$
$$S_2 = \sigma(n_{02}n_{03} - n_{01}n_{04}) + \alpha(\rho_0 + \bar{\eta})(n_{02} - \rho_0\kappa_2). \tag{B.5}$$

Appendix B.2.

Linearizing around (0) we get a 4×4 X_ρ, X_j, X_{23}, X_2 linear system with four linear differential operators and write Ω_2 as follows:

$$\sum_{i=1}^{i=4}\Omega_i = \begin{vmatrix} \partial_t & \partial_x & 0 & 0 \\ \partial_x - A_\rho & \partial_t + A_J & 3/2\partial_x & -3\partial_x \\ -A_2 & 0 & \partial_t - \partial_x/2 + A_J & \partial_x \\ -A_3 & -A_4 & -A_5 & \partial_t + \partial_x/2 + A_1 \end{vmatrix} = 0$$

$$\begin{aligned} &A_1 = A_J + \sigma(2n^+_{0,14} + n^+_{0,23}) \qquad A_2 = \alpha(2\rho_0\kappa_{2,3} - n^-_{0,23}) + \eta\kappa_{2,3} \\ &A_3 = \alpha(2\rho_0\kappa_2 - n_{02}) + \eta\kappa_2 + \sigma n^+_{0,14}/2 \qquad A_4 = \sigma n^-_{0,41}/2 \\ &A_5 = \sigma(n_{04} + 2n_{02} + 3n_{01})/2 \\ &\Omega_2 := \partial^2_{t^2}(A_J(A_J + 2A_1) + \partial^2_{xt}[A_\rho(A_1 + A_J) + A_J(A_5 - 3A_4 + (A_J - A_1)/2)] \\ &\qquad +\partial^2_{x^2}[A_J(A_\rho/2 - A_1 + 3A_3) + A_5(A_\rho + 3A_2) - A_1(A_\rho/2 + 3A_2/2)]. \end{aligned} \tag{B.6}$$

Appendix C. Shock waves for the square $8v_i$ model (7.1)

Appendix C.1.

Compatible Riccati solutions with five $N_i, i = 1, \ldots, 5$ evolution equations:

$$(\partial_t + e_i\partial_x)N_i = \sigma_{1,i}C_1 + \sigma_{2,i}C_2 + (\alpha\rho + \beta)(\rho\kappa_i - N_i) + S_i \tag{C.1}$$

with C_1, C_2 in (7.1), $e_i = 1, 1, 0, -1, -1$, $\sigma_{1,i}/\sigma_1 = -1, 0, 1, 0, -1$, $\sigma_{2,i}/\sigma_2 = 2, -1, 0, 1, 2$. We define, as in (A.9), $\bar{n}^\pm_{ij}, \bar{\sigma}_i = \sigma_i/\alpha, \bar{\gamma}, \bar{n}_{ijkl}, n^\pm_{0,ij} = n_{0i} \pm n_{0j} \ldots$:

$$C_{Lax} = 1 - \xi^2 - 2\bar{n}_3 \qquad (\rho_1, j_1) \to 1 = \bar{n}^+_{1,5} + 2(\bar{n}^+_{2,3} + \bar{n}_3) \qquad \xi = \bar{n}^-_{1,5} + 2\bar{n}^-_{2,4}. \tag{C.2}$$

For the $J, N^+_{2,4}, N_2, N_3$ equations we write the four compatible $\bar{\gamma}$ and (2.5*a*)–(2.5*c*) relations:

$$\begin{aligned} \bar{\gamma} &= (\kappa_J - \xi)/C_{Lax} = (\kappa^+_{2,4} - \bar{n}^+_{2,4})/(\bar{n}^-_{2,4} - \xi\bar{n}^+_{2,4}) \\ &= (\bar{n}_3 - \kappa_3 - \bar{\sigma}_1\bar{n}_{1533})/\xi\bar{n}_3 = (\bar{n}_2 - \kappa_2 - \bar{\sigma}_2\bar{n}_{1425})/\bar{n}_2(\xi - 1) \end{aligned} \tag{C.3}$$

$$\begin{aligned} &\xi A_\rho = \kappa_J B_\rho - j_0 \qquad \bar{n}^+_{2,4}A_\rho = \kappa^+_{2,4}B_\rho - n^+_{0,24} \\ &\bar{n}_3 A_\rho = \kappa_3 B_\rho - n_{03} + \bar{\sigma}_1\bar{n}_{0,1533} \qquad \bar{n}_2 A_\rho = \kappa_2 B_\rho - n_{02} + \bar{\sigma}_2\bar{n}_{0.1425}. \end{aligned} \tag{C.4}$$

Appendix C.2.

We linearize around (0) with a 5×5 $X_\rho, X_j, X_{2,4}, X_2, X_3$ and five Ω_i:

$$\begin{aligned} &A_2 = \alpha(2\rho_0\kappa_{24} - n^+_{0,24}) + \eta\kappa_{24} \qquad \tilde{A}_2 = A_J + \sigma_2(n^+_{0,15} + 2n^+_{0,24}) \\ &A_3 = \alpha(2\rho_0\kappa_2 - n_{02}) + \eta\kappa_2 + \sigma_2 n^-_{0,42}/2 \qquad \tilde{A}_3 = A_J + \sigma_1(n^+_{0,15} + 2n_{03}) \\ &A_4 = \sigma_2 n^+_{0,24}/2 \\ &A_5 = \sigma_2(n_{01} + 2n_{02}) \qquad A_6 = \sigma_2 n^-_{0,24} \qquad A_7 = \alpha(2\rho_0\kappa_3 - n_{03}) + \eta\kappa_3 + \sigma_1 n^+_{0,15}/2 \\ &A_8 = \sigma_1 n^-_{0,51}/2 = -A_{10}/4 \qquad A_9 = -2\sigma_1 n_{01} \\ &\sum_{i=1}^{i=5}\Omega_i = \begin{vmatrix} \partial_t & \partial_x & 0 & 0 & 0 \\ \partial_x - A_\rho & \partial_t + A_J & 0 & 0 & -2\partial_x \\ -A_2 & 0 & \partial_t - \partial_x + A_J & 2\partial_x & 0 \\ -A_3 & -A_4 & -A_5 & \partial_t + \partial_x + \tilde{A}_2 & 0 \\ -A_7 & -A_8 & -A_9 & -A_{10} & \partial_t + \tilde{A}_3 \end{vmatrix} = 0. \end{aligned} \tag{C.5}$$

Appendix D. Shock waves for the square $9v_i$ model (7.2), (7.3)

Appendix D.1.

Compatible Riccati solutions for $J, N_{240}, N_0, N^+_{2,3}, N^+_{3,4}$ with $\bar{n}_{240} = \bar{n}^+_{2,4} - \bar{n}_0/2$:

$$C_{Lax} = 1 - \xi^2 - 2\bar{n}_3 - \bar{n}_0 \qquad (\rho_1, j_1) \rightarrow 1 = \bar{n}^+_{1,5} + 2(\bar{n}^+_{2,4} + \bar{n}_3) + \bar{n}_0 \\ \xi = \bar{n}^-_{1,5} + 2\bar{n}^-_{2,4}. \tag{D.1}$$

We write the five compatible $\bar{\gamma} C_{Lax} = \kappa_J - \xi$ and the (2.5a)–(2.5c) relations without S_i, S_J:

$$\begin{aligned} \bar{\gamma} &= (\kappa_{240} - \bar{n}_{240})/(\bar{n}^-_{24} - \xi\bar{n}_{240}) = (\bar{n}_0 - \kappa_0 - 2\bar{\sigma}_0(\bar{n}_{1320} + \bar{n}_{5340})/\bar{n}_0\xi \\ &= (\bar{n}^+_{3,4} - \kappa^+_{3,4} - \bar{\sigma}_1\bar{n}_{1533} + \bar{\sigma}_0\bar{n}_{1320} + \bar{\sigma}_2\bar{n}_{1425}/(\xi\bar{n}^+_{3,4} + \bar{n}_4) \\ &= (\bar{n}^+_{2,3} - \kappa^+_{2,3} - \bar{\sigma}_2\bar{n}_{1425} - \bar{\sigma}_1\bar{n}_{1533} + \bar{\sigma}_0\bar{n}_{5340}/(\bar{n}^+_{2,3}\xi - \bar{n}_2) \\ \kappa_{240} &= \kappa^+_{2,4} - \kappa_0/2 \end{aligned} \tag{D.2}$$

$$\begin{aligned} &\bar{n}_{240}A_\rho = \kappa_{240}B_\rho - n^+_{0,24} + n_{00}/2 \qquad \bar{n}_0 A_\rho = \kappa_0 B_\rho - n_{00} + 2\bar{\sigma}_0(\bar{n}_{0,1320} + \bar{n}_{0,5340}) \\ &\bar{n}^+_{3,4}A_\rho = \kappa^+_{3,4}B_\rho - n^+_{0,34} + \bar{\sigma}_1\bar{n}_{0,1533} + \bar{\sigma}_2\bar{n}_{0,1425} - \bar{\sigma}_0\bar{n}_{0,1320} \qquad \xi A_\rho = \kappa_J B_\rho - j_0 \\ &\bar{n}^+_{2,3}A_\rho = \kappa^+_{2,3}B_\rho - n^+_{0,23} - \bar{\sigma}_2\bar{n}_{0,1425} - \bar{\sigma}_1\bar{n}_{0,1533} - \bar{\sigma}_0\bar{n}_{0,5340} \qquad \bar{\sigma}_i = \sigma_i/\alpha. \end{aligned} \tag{D.3}$$

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